

ONE-PARAMETER CONTRACTIONS OF LIE-POISSON BRACKETS

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1. INTRODUCTION

Let \mathfrak{q} be a finite-dimensional Lie algebra defined over a field \mathbb{K} of characteristic zero. Then the symmetric algebra $\mathcal{S}(\mathfrak{q}) = \mathbb{K}[\mathfrak{q}^*]$ carries a Poisson structure induced by the Lie bracket on \mathfrak{q} . In this paper, we study the algebra $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ of symmetric invariants. By a theorem of Duflo, it is isomorphic to the centre $ZU(\mathfrak{q})$ of the universal enveloping algebra $U(\mathfrak{q})$, and therefore is of much interest in representation theory. We can also say that $\mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$ coincides with the Poisson centre $Z\mathcal{S}(\mathfrak{q})$ of $\mathcal{S}(\mathfrak{q})$ (for the definition of this object see Section 2), and one can employ methods of Poisson geometry to investigate this algebra.

To be more precise, the Lie algebra in question is a contraction of some other Lie algebra, whose symmetric invariants are well understood. Already contractions of simple (non-Abelian) Lie algebras provide a fairly interesting and not yet completely explored field of research. Let $\mathfrak{f} \subset \mathfrak{q}$ be a Lie subalgebra and $V \subset \mathfrak{q}$ a complementary subspace, not necessarily \mathfrak{f} -stable. Then one contracts \mathfrak{q} to a Lie algebra $\tilde{\mathfrak{q}} = \mathfrak{f} \ltimes V$, where V is an Abelian ideal and the action of \mathfrak{f} on it comes from the projection $\text{pr}_V : \mathfrak{q} \rightarrow V$ along \mathfrak{f} . A more sophisticated description of contractions of Poisson and Lie algebras is given in Section 3.

Suppose that \mathfrak{g} is a reductive Lie algebra. Let F_1, \dots, F_ℓ with $\ell = \text{rk } \mathfrak{g}$ be homogeneous generators of $Z\mathcal{S}(\mathfrak{g})$. Then, by [K, Theorem 9], their differentials $d_\xi F_i$ at a point $\xi \in \mathfrak{g}^*$ are linear independent if and only if $\dim \mathfrak{g}_\xi = \ell$ for the stabiliser in the coadjoint action. This is known as Kostant's regularity criterion. For an arbitrary Lie algebra \mathfrak{q} , the notion of *index*, $\text{ind } \mathfrak{q} = \min_{\xi \in \mathfrak{q}^*} \dim \mathfrak{q}_\xi$, generalises the rank in the reductive case. A Lie algebra \mathfrak{q} of index ℓ is said to be of *Kostant type*, if $Z\mathcal{S}(\mathfrak{q})$ is freely generated by homogeneous polynomials H_1, \dots, H_ℓ such that they give Kostant's regularity criterion on \mathfrak{q}^* . Set $\mathfrak{q}_{\text{sing}}^* := \{\xi \in \mathfrak{q}^* \mid \dim \mathfrak{q}_\xi > \text{ind } \mathfrak{q}\}$. We say that \mathfrak{q} has a “codim-2” property or satisfies a “codim-2” condition, if $\dim \mathfrak{q}_{\text{sing}}^* \leq \dim \mathfrak{q} - 2$. The importance of this condition was first noticed in [PPY] and [P07].

The decomposition $\mathfrak{q} = \mathfrak{f} \oplus V$ induces a bi-grading on $\mathcal{S}(\mathfrak{q})$. For each homogeneous $F \in \mathcal{S}(\mathfrak{q})$, let $\deg_t F$ denote its degree in V and F^\bullet the bi-homogeneous component of F of bi-degrees $(\deg F - \deg_t F, \deg_t F)$ in \mathfrak{f} and V , respectively. In case $\mathfrak{q} = \mathfrak{g}$ is reductive and $\tilde{\mathfrak{g}}$ is the contraction of \mathfrak{g} corresponding to the decomposition $\mathfrak{g} = \mathfrak{f} \oplus V$, a simplification of our main result, Theorem 3.8, can be formulated as follows.

Suppose that $\text{ind } \tilde{\mathfrak{g}} = \ell = \text{rk } \mathfrak{g}$. Then

- $\sum \deg_t F_i \geq \dim V$ and the polynomials F_i^\bullet are algebraically independent if and only if $\sum \deg_t F_i = \dim V$.
- Moreover, if the equality holds and the polynomials F_i^\bullet generate $Z\mathcal{S}(\tilde{\mathfrak{g}})$ (this can be guaranteed by the “codim-2” property of $\tilde{\mathfrak{g}}$), then $\tilde{\mathfrak{g}}$ is of Kostant type.

The proof of Theorem 3.8 relies on Lemma 2.1, a statement about Poisson brackets in the algebraic setting, and good behaviour of the Poisson tensor under contractions. The resulting algebras $\tilde{\mathfrak{g}}$ are non-reductive and there is no general method for describing their symmetric invariants.

Note that Theorem 3.8 is stated and proved for arbitrary polynomial Poisson algebras that are not necessary symmetric algebras of any finite-dimensional \mathfrak{q} . We do not consider applications of the more general version in this paper, but hope to explore this subject in the (near) future.

Two types of contractions $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ are studied here. In both cases it is assumed that the ground field \mathbb{K} is algebraically closed. The first contraction comes from a \mathbb{Z}_2 -grading (or symmetric decomposition) $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ of \mathfrak{g} . It was conjectured by D. Panyushev [P07], that in this setting $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial algebra in ℓ variables. As was shown in [P07], $\text{ind } \tilde{\mathfrak{g}} = \ell$ and $\tilde{\mathfrak{g}}$ has the “codim-2” property. Also for many \mathbb{Z}_2 -gradings the polynomiality of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ was established in that paper of Panyushev. For four of the remaining cases, we construct homogeneous generators F_i such that $\sum \deg_t F_i \leq \dim \mathfrak{g}_1$. Since also $\sum \deg_t F_i \geq \dim \mathfrak{g}_1$ by Theorem 3.8, we get the equality $\sum \deg_t F_i = \dim \mathfrak{g}_1$ and thereby prove that their components F_i^\bullet freely generate $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$. This line of argument resemblances proofs of [PPY, Theorems 4.2&4.4]. Our result confirms a weaker version of Panyushev’s conjecture. If the restriction homomorphism $\mathbb{K}[\mathfrak{g}]^{\mathfrak{g}} \rightarrow \mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ is surjective, then $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial algebra in ℓ variables, see Theorem 4.5.

The second contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ was recently introduced by E. Feigin [F10] and for the resulting Lie algebra, $\tilde{\mathfrak{g}}$ -invariants in $\mathcal{S}(\tilde{\mathfrak{g}})$ and $\mathbb{K}[\tilde{\mathfrak{g}}]$ were studied in [PY]. Here the decomposition is $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where $\mathfrak{b} = \text{Lie } B$ is a Borel subalgebra and \mathfrak{n}^- is the nilpotent radical of an opposite Borel. Complementing and relying on results of [PY], we show that $\tilde{\mathfrak{g}}$ is of Kostant type (Lemma 5.2), compute its fundamental semi-invariant (see Definition 5.4 and Theorem 5.5), and prove that the subalgebra $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} \subset \mathcal{S}(\tilde{\mathfrak{g}})$ generated by semi-invariants of $\tilde{\mathfrak{g}}$ (Definition 4.7) is a polynomial algebra in 2ℓ variables, Theorem 5.8. If \mathfrak{g} is not of type A , then $\tilde{\mathfrak{g}}$ does not have the “codim-2” property. However, the quotient map $\mathbb{K}[\tilde{\mathfrak{g}}^*] \rightarrow \mathbb{K}[\tilde{\mathfrak{g}}^*]^{\tilde{\mathfrak{g}}}$ is equidimensional and $U(\tilde{\mathfrak{g}})$ is a free $ZU(\tilde{\mathfrak{g}})$ -module [PY].

Finally, Section 5.2 contains a few observations related to subregular orbital varieties \mathcal{D}_i , which are linear subspaces of \mathfrak{n} of codimension 1 forming the complement of the open B -orbit in \mathfrak{n} . In particular, in Proposition 5.12, we list all \mathcal{D}_i such that the stabiliser B_x is Abelian for a generic $x \in \mathcal{D}_i$.

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2. GENERALITIES ON POLYNOMIAL POISSON STRUCTURES

In this section, we recall a rather important equality in Poisson algebras, which has an origin in mathematical physics [OR].

Let \mathbb{K} be a field of characteristic zero and $\mathbb{A}^n = \mathbb{A}_{\mathbb{K}}^n$ the n -dimensional affine space with the algebra of regular functions $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]$. Let Ω be the algebra of regular, i.e., with polynomial coefficients, differential forms on \mathbb{A}^n and W the algebra of derivations of \mathcal{A} . Both are free \mathcal{A} -modules with bases consisting of skew-monomials in dx_i and $\partial_i = \partial_{x_i}$, respectively. In other words, W is a graded skew-symmetric algebra generated by polynomial vector fields on \mathbb{A}^n . We identify Ω^0 with \mathcal{A} and regard Ω^1 as the \mathcal{A} -module of global sections of the cotangent bundle $T^*\mathbb{A}^n$. Let W^1 be an \mathcal{A} -module generated by ∂_i with $1 \leq i \leq n$. We view the exterior powers $\Omega^k = \Lambda_{\mathcal{A}}^k \Omega^1$ and $W^k := \Lambda_{\mathcal{A}}^k W^1$ as dual \mathcal{A} -modules by extending the canonical non-degenerate \mathcal{A} -pairing $dx_i(\partial_j) = \delta_{ij}$.

Let $\omega = dx_1 \wedge \dots \wedge dx_n$ be the volume form. If f and g are elements of Ω^k and Ω^{n-k} , respectively, then $f \wedge g = a\omega$ with $a \in \mathcal{A}$. We will say that in this situation $a = (f \wedge g)/\omega$ and f/ω is an element of $(\Omega^{n-k})^*$ such that $(f/\omega)(g) = a$. This defines an \mathcal{A} -linear map

$$\frac{1}{\omega} : \Omega^k \rightarrow (\Omega^{n-k})^* \cong W^{n-k}.$$

Suppose that \mathcal{A} possesses a Poisson structure $\{ , \} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ and let π denote the corresponding *Poisson tensor (bivector)*, the element of $\text{Hom}_{\mathcal{A}}(\Omega^2, \mathcal{A})$ satisfying $\pi(df \wedge dg) = \{f, g\}$ for all $f, g \in \mathcal{A}$. (It is *not* assumed that the coefficients of π are linear functions.) In view of the duality between forms and vector fields, we may regard π as an element of W^2 . For $\xi \in \mathbb{A}^n$, π_{ξ} can be viewed as a skew-symmetric matrix with entries $\{x_i, x_j\}(\xi)$. The *index* of the Poisson algebra \mathcal{A} , denoted $\text{ind } \mathcal{A}$, is defined as

$$\text{ind } \mathcal{A} := n - \text{rk } \pi, \text{ where } \text{rk } \pi = \max_{\xi \in \mathbb{A}^n} \text{rk } \pi_{\xi}.$$

An element $a \in \mathcal{A}$ is said to be *central*, if $\{a, \mathcal{A}\} = 0$. Correspondingly, the set $Z\mathcal{A} = Z(\mathcal{A}, \pi)$ of all central elements is called the *Poisson centre* of \mathcal{A} .

Set $\text{Sing } \pi := \{\xi \in \mathbb{A}^n \mid \text{rk } \pi_{\xi} < \text{rk } \pi\}$. Clearly, $\text{Sing } \pi$ is a proper Zariski closed subset of \mathbb{A}^n . By definition, $\pi(da \wedge db) = 0$ for all $a \in Z\mathcal{A}$ and all $b \in \mathcal{A}$. Hence the linear subspace $\{d_{\xi}a \mid a \in Z\mathcal{A}\}$ lies in the kernel of π_{ξ} and we have

$$\text{tr. deg}_{\mathbb{K}} Z\mathcal{A} \leq \text{ind } \mathcal{A}.$$

For $g_1, \dots, g_m \in \mathcal{A}$, the *Jacobian locus* $\mathcal{J}(g_1, \dots, g_m)$ consists of all $\xi \in \mathbb{A}^n$ such that the differentials $d_{\xi}g_1, \dots, d_{\xi}g_m$ are linearly dependent. In other words, $\xi \in \mathcal{J}(g_1, \dots, g_m)$ if and only if $(dg_1 \wedge \dots \wedge dg_m)_{\xi} = 0$. The set $\mathcal{J}(g_1, \dots, g_m)$ is Zariski closed in \mathbb{A}^n and it coincides with \mathbb{A}^n if and only if g_1, \dots, g_m are algebraically dependent.

Given $k \in \mathbb{N}$ we let

$$\Lambda^k \pi := \underbrace{\pi \wedge \pi \wedge \dots \wedge \pi}_{k \text{ factors}},$$

be an element of W^{2k} . Note that $\Lambda^k \pi \neq 0$ if and only if π_ξ contains a non-zero $2k \times 2k$ -minor for some $\xi \in \mathbb{A}^n$. Therefore $\Lambda^k \pi = 0$ for $k > (\text{rk } \pi)/2$ and $\Lambda^k \pi \neq 0$ for $k \leq (\text{rk } \pi)/2$. The following statement can be extracted from the proofs of [OR, Theorem 3.1], [PPY, Theorem 1.2], [P07, Theorem 1.2].

Lemma 2.1. *Let $\mathcal{A} = \mathbb{K}[x_1, \dots, x_n]$ be a Poisson algebra of index ℓ and let $\{F_1, \dots, F_\ell\} \subset Z\mathcal{A}$ be a set of algebraically independent polynomials. Then there are coprime $q_1, q_2 \in \mathcal{A} \setminus \{0\}$ such that*

$$q_1 \frac{dF_1 \wedge \dots \wedge dF_\ell}{\omega} = q_2 \Lambda^{(n-\ell)/2} \pi.$$

Proof. Set $\mathcal{F} = dF_1 \wedge \dots \wedge dF_\ell$. Because of the inequality: $\text{tr. deg } Z(\mathcal{A}) \leq \ell$, the polynomials F_1, \dots, F_ℓ , and F are algebraically dependent for each $F \in Z\mathcal{A}$ and therefore $\mathcal{F} \wedge F = 0$. Clearly $\pi(dF, \cdot) = 0$ and hence $\Lambda^{n-\ell/2} \pi$ is zero on $dF \wedge \Omega^{n-\ell-1}$.

Changing the ordering of the coordinates, we may assume that $\mathcal{F} \wedge dx_1 \wedge \dots \wedge dx_{n-\ell} \neq 0$. Let $\xi \in \mathbb{A}^n$ be such that $\text{rk } \pi_\xi = n - \ell$ and the elements $d_\xi F_i$ together with $\{dx_j \mid j \leq n - \ell\}$ form a basis of $T_\xi^* \mathbb{A}^n$. Then

$$\Lambda^{n-\ell}(T_\xi^* \mathbb{A}^n) = \text{Ker} \left(\frac{\mathcal{F}}{\omega} \right)_\xi \oplus \mathbb{K}(dx_1 \wedge \dots \wedge dx_{n-\ell}).$$

Note that here the kernel of $(\mathcal{F}/\omega)_\xi$ lies also in the kernel of $\Lambda^{(n-\ell)/2} \pi_\xi$. Next $\Lambda^{(n-\ell)/2} \pi_\xi \neq 0$. Therefore $(\mathcal{F}/\omega)_\xi$ is proportional to $\Lambda^{(n-\ell)/2} \pi_\xi$. We can conclude that \mathcal{F}/ω and $\Lambda^{(n-\ell)/2} \pi$ are proportional on an open subset of \mathbb{A}^n . It follows that there exist non-zero coprime $q_1, q_2 \in \mathcal{A}$ such that $q_1(\mathcal{F}/\omega) = q_2 \Lambda^{(n-\ell)/2} \pi$. \square

Of particular interest are situations where $q_1, q_2 \in \mathbb{K}$ for q_1, q_2 as above. This can be guarantied by “codim-2” conditions, see e.g. [PPY, Theorem 1.2]. If $\dim \text{Sing } \pi \leq n - 2$, then q_1 must be a scalar. If $\dim \mathcal{J}(F_1, \dots, F_\ell) \leq n - 2$, then q_2 must be a scalar.

In case π is homogeneous, i.e., all the (polynomial) coefficients of π are of the same degree, we can say that $\deg \Lambda^k \pi = k \deg \pi$. Suppose that π and all the F_i ’s are homogeneous. Then q_1, q_2 are also homogeneous and

$$\deg q_1 - \ell + \sum_{i=1}^{\ell} \deg F_i = \deg q_2 + \frac{n-\ell}{2} \deg \pi.$$

If, for example, $2 \sum \deg F_i = 2\ell + (n - \ell) \deg \pi$, then $\deg q_1 = \deg q_2$ and knowing that q_1 is constant, we also know that q_2 is a constant.

Poisson tensors of degree 1 correspond to finite-dimensional Lie algebras \mathfrak{q} over \mathbb{K} . In this case $\mathbb{A}^n = \mathfrak{q}^*$ is the dual space of an n -dimensional Lie algebra \mathfrak{q} and $\mathcal{A} = \mathcal{S}(\mathfrak{q}) = \mathbb{K}[\mathfrak{q}^*]$

is the symmetric algebra of \mathfrak{q} . Set $\text{ind } \mathfrak{q} = \text{ind } \mathcal{S}(\mathfrak{q})$ and $\mathfrak{q}_{\text{sing}}^* = \text{Sing } \pi$. Note that

$$\text{ind } \mathfrak{q} = \min_{\gamma \in \mathfrak{q}^*} \dim \mathfrak{q}_\gamma = \dim \mathfrak{q}_\alpha \text{ for all } \alpha \in \mathfrak{q}_{\text{reg}}^* = \mathfrak{q}^* \setminus \mathfrak{q}_{\text{sing}}^*.$$

It is also worth mentioning that $Z\mathcal{S}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$.

Suppose that \mathfrak{g} is a non-Abelian reductive Lie algebra, then $\text{ind } \mathfrak{g} = \text{rk } \mathfrak{g}$, the algebra of symmetric invariants $Z\mathcal{S}(\mathfrak{g})$ is freely generated by homogeneous polynomials F_1, \dots, F_ℓ with $\ell = \text{rk } \mathfrak{g}$, and $2 \sum \deg F_i = n + \ell$. Moreover, $\dim \mathfrak{g}_{\text{sing}}^* = n - 3$. Therefore, after a suitable renormalisation,

$$(2.1) \quad \frac{dF_1 \wedge \dots \wedge dF_\ell}{\omega} = \Lambda^{(n-\ell)/2} \pi.$$

This is known as Kostant's regularity criterion: $x \in \mathfrak{g}_{\text{reg}}^*$ if and only if the differentials $d_x F_i$ are linear independent, [K, Theorem 9].

Definition 2.2. Equation (2.1) is called the *Kostant equality* and we will say that a Poisson algebra \mathcal{A} (or a Lie algebra \mathfrak{q}) is of *Kostant type*, if $Z\mathcal{A}$ is generated by ℓ polynomials satisfying the Kostant equality.

Apart from reductive and Abelian Lie algebras, examples of Lie algebras of Kostant type are provided by the centralisers \mathfrak{g}_e of nilpotent elements in \mathfrak{sl}_m and \mathfrak{sp}_{2m} [PPY], truncated seaweed (biparabolic) subalgebras of \mathfrak{sl}_m and \mathfrak{sp}_{2m} [J], and semi-direct products related to symmetric decompositions $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, see [P07] and Section 4 here.

Let (e, h, f) be an \mathfrak{sl}_2 -triple in $\mathfrak{g} = \text{Lie } G$. Then $\mathbf{S}_e = e + \mathfrak{g}_f$ is the *Slodowy slice* of Ge at e and $\mathbb{K}[\mathbf{S}_e]$ inherits a Poisson bracket from $\mathfrak{g}(\cong \mathfrak{g}^*)$. These Poisson algebras are of Kostant type and they are associated graded algebras of the finite W -algebras, see [Pr] and [PPY, Section 2]. Note that all mentioned above Poisson and Lie algebras of Kostant type have the “codim-2” property.

3. CONTRACTIONS OF POISSON TENSORS

We begin with a definition of a contraction in the Lie algebra setting. Let \mathfrak{q} be a Lie algebra, $\mathfrak{f} \subset \mathfrak{q}$ a Lie subalgebra, and $V \subset \mathfrak{q}$ a complimentary (to \mathfrak{f}) subspace. We do not require V to be \mathfrak{f} -stable. For each $t \in \mathbb{K}^\times$, let $\varphi_t : \mathfrak{q} \rightarrow \mathfrak{q}$ be a linear map multiplying vectors in V by t and vectors in \mathfrak{f} by 1. These automorphisms form a one-parameter subgroup in $\text{GL}(\mathfrak{q})$. Each φ_t defines a new Lie algebra structure $[\cdot, \cdot]_t$ on the same vector space \mathfrak{q} . Let $\text{pr}_{\mathfrak{f}}$ and pr_V be the projection on \mathfrak{f} and V , respectively. Then

$$[\xi, \eta]_t = [\xi, \eta], \quad [\xi, v]_t = t \text{pr}_{\mathfrak{f}}([\xi, v]) + \text{pr}_V([\xi, v]), \quad [v, w]_t = t^2 \text{pr}_{\mathfrak{f}}([v, w]) + t \text{pr}_V([v, w]),$$

for $\xi, \eta \in \mathfrak{f}$, $v, w \in V$. We can pass to the limit $\lim_{t \rightarrow 0} [\cdot, \cdot]_t$ and get yet another Lie algebra structure on \mathfrak{q} . Let $\tilde{\mathfrak{q}}$ denote this contraction of \mathfrak{q} . Then $\tilde{\mathfrak{q}} = \mathfrak{f} \ltimes V$, where V is an Abelian ideal of $\tilde{\mathfrak{q}}$ and the action of \mathfrak{f} on V is given by pr_V . (The reader feeling uncomfortable with

taking the limit, although it can be defined in a purely algebraic setting, may assume that t takes values in $\mathbb{Q} \subset \mathbb{K}$.)

In a coordinate free way, the t -commutator $[\cdot, \cdot]_t$ or the t -Poisson bracket $\{\cdot, \cdot\}_t$ is defined by

$$\{x, y\}_t = \varphi_t^{-1}(\{\varphi_t(x), \varphi_t(y)\})$$

for $x, y \in \mathfrak{q}$. Extending φ_t to the symmetric algebra $\mathcal{S}(\mathfrak{q})$ as well as to W and Ω , one can say that $\pi_t = \varphi_t^{-1}(\pi)$. Let \mathfrak{q}_t stand for the Lie algebra corresponding to π_t . Then the Poisson centre of $\mathcal{S}(\mathfrak{q}_t)$ equals $\varphi_t^{-1}(Z\mathcal{S}(\mathfrak{q}))$.

For $H \in \mathcal{S}(\mathfrak{q})$, let $\deg_t H$ be the degree in t of $\varphi_t(H)$. This means that $\varphi_t(H) = t^d H_d + t^{d-1} H_{d-1} + \dots + H_0$, where $d = \deg_t H$, $H_i \in \mathcal{S}(\mathfrak{q})$, and $H_d \neq 0$. We will say that $H^\bullet := H_d$ is the highest (t -) component of H .

Lemma 3.1. *If $H \in Z\mathcal{S}(\mathfrak{q})$, then H^\bullet is a central element in $\mathcal{S}(\tilde{\mathfrak{q}})$.*

Proof. Since $H \in \mathcal{S}(\mathfrak{q})^{\mathfrak{q}}$, its preimage $\varphi_t^{-1}(H)$ is a central element in $\mathcal{S}(\mathfrak{q}_t)$, which one can write as $\varphi_t^{-1}(H) = t^{-d} H_d + t^{1-d} H_{d-1} + \dots + t^{-1} H_1 + H_0$. Multiplying it by t^d , we get that $\sum_{j=0}^d t^{d-j} H_j$ is also a central element in $\mathcal{S}(\mathfrak{q}_t)$. Passing to the limit at $t \rightarrow 0$, one obtains that $H_d = H^\bullet$ is an element of $Z\mathcal{S}(\tilde{\mathfrak{q}})$. \square

The automorphism $\varphi_t : \mathfrak{q} \rightarrow \mathfrak{q}$ does not need to be of degree 1 in t as well as the Poisson tensor π does not need to be linear. We can consider a one-parameter family of linear automorphisms of \mathbb{A}^n and the corresponding deformation of Poisson structures on it. The only important thing is that there exists a limit $\lim_{t \rightarrow 0} \pi_t$. In order to be consistent with the Lie algebra case, we identify \mathbb{A}^n with \mathbb{K}^n . Let φ be a \mathbb{K} -linear automorphism of the dual space $(\mathbb{K}^n)^*$. Then φ extends to \mathbb{K} -linear automorphisms of \mathbb{A}^n , $\mathcal{A} = \mathbb{K}[\mathbb{A}^n]$, W , and Ω .

Definition 3.2. Let π be a polynomial Poisson tensor on $\mathbb{A}^n \cong \mathbb{K}^n$. Suppose that we have a family of automorphisms φ_t given by a regular map $\mathbb{K}^\times \rightarrow \text{GL}((\mathbb{K}^n)^*)$ and that the formal expression of $\pi_t := \varphi_t^{-1}(\pi)$ is an element of $W^2[t]$. Then $\tilde{\pi} := \lim_{t \rightarrow 0} \pi_t$ is called a *contraction* of π . For each $H \in \mathcal{A}$, we define its *highest (t -) component* as a non-zero polynomial H^\bullet such that $H^\bullet = \lim_{t \rightarrow 0} t^d \varphi_t^{-1}(H)$ for some $d =: \deg_t H$. (One readily sees the uniqueness of this d .)

Lemma 3.3. *If $\tilde{\pi}$ is a contraction of π , then $\tilde{\pi}$ is again a Poisson tensor and for each $H \in Z(\mathcal{A}, \pi)$, the polynomial H^\bullet is an element of $Z(\mathcal{A}, \tilde{\pi})$.*

Proof. An element $R \in W^2$ is a Poisson tensor if and only if $[R, R] = 0$. (This is a way to state the Jacobi identity.) Since $[\pi_t, \pi_t] = 0$ for all non-zero t and $\pi_t \in W^2[t]$, we have $\pi_t = \tilde{\pi} + tR$, where $R \in W^2[t]$, and $0 = [\pi_t, \pi_t] = [\tilde{\pi}, \tilde{\pi}] + t\tilde{R}$ with $\tilde{R} \in W^2[t]$. Therefore $[\tilde{\pi}, \tilde{\pi}] = 0$.

In order to prove the second statement one repeats the argument of Lemma 3.1. \square

Example 3.4. Suppose we have a decomposition $\mathfrak{q} = V_0 \oplus V_1 \oplus \dots \oplus V_{m-1}$, where V_0 is a subalgebra and in general $[V_i, V_j] \subset \bigoplus_{k \leq i+j} V_k$. Then one can define $\varphi_t : \mathfrak{q} \rightarrow \mathfrak{q}$ by setting $\varphi|_{V_j} = t^j \text{id}$ and consider Lie algebra structures $[\cdot, \cdot]_t$ on \mathfrak{q} . Clearly, there is a limit at $t \rightarrow 0$ and the resulting Lie algebra $\tilde{\mathfrak{q}}$ has a \mathbb{Z} -grading with at most m non-zero components.

Contractions of Lie algebras as in Example 3.4 were studied by Panyushev [P09].

Definition 3.5. Let $\ell = \text{ind } \mathfrak{q}$. We say that a set $\{H_1, \dots, H_\ell\} \subset \mathcal{S}(\mathfrak{q})^\mathfrak{q}$ is a *good generating system* with respect to a contraction $[\cdot, \cdot]_t \rightsquigarrow [\cdot, \cdot]_{\tilde{\mathfrak{q}}}$ if the polynomials H_i generate $\mathcal{S}(\mathfrak{q})^\mathfrak{q}$ and their highest components H_i^\bullet are algebraically independent.

Let $(Z\mathcal{S}(\mathfrak{q}))^\bullet$ be the algebra of highest components of $Z\mathcal{S}(\mathfrak{q})$, i.e., this is an algebra generated by H^\bullet with $H \in Z\mathcal{S}(\mathfrak{q})$.

Lemma 3.6. *If $\{H_1, \dots, H_\ell\} \subset \mathcal{S}(\mathfrak{q})^\mathfrak{q}$ is a good generating system, then H_i^\bullet generate $(Z\mathcal{S}(\mathfrak{q}))^\bullet$.*

Proof. Each non-zero element $g \in Z\mathcal{S}(\mathfrak{q})$ can be expressed as a polynomial P in H_i . Suppose that P is a sum $P = \sum_{\bar{s}} a_{\bar{s}} H_1^{s_1} \dots H_\ell^{s_\ell}$ over some $\bar{s} \in \mathbb{Z}_{\geq 0}^\ell$. Define \tilde{P} as a sum of those monomials (with the coefficients $a_{\bar{s}}$), where the degree in t , $s_1 \deg_t H_1 + \dots + s_\ell \deg_t H_\ell$, is maximal. Then $\tilde{P}(H_1^\bullet, \dots, H_\ell^\bullet)$ is a non-zero polynomial, because the elements H_i^\bullet are algebraically independent, and it equals g^\bullet by the construction. \square

3.1. Contractions and the Kostant equality.

Example 3.7. Suppose that $\mathfrak{q} = \mathfrak{sl}_2$ and a contraction $\mathfrak{q} \rightsquigarrow \tilde{\mathfrak{q}}$ is defined by a decomposition $\mathfrak{sl}_2 = \mathfrak{so}_2 \oplus V$, where V is an \mathfrak{so}_2 -invariant complement. In a standard basis $\{e, h, f\}$ the automorphism φ_t multiplies e and f by t . In the basis $\{e/t, h, f/t\}$ the Poisson tensor π_t is given by the same formula as π in the original basis. Therefore we have

$$\frac{dF}{d(e/t) \wedge dh \wedge d(f/t)} = h \partial_{e/t} \wedge \partial_{f/t} + 2 \frac{e}{t} \partial_h \wedge \partial_{e/t} + 2 \frac{f}{t} \partial_{f/t} \wedge \partial_h,$$

where F is a suitably normalised invariant of degree 2, explicitly $F = -\frac{h^2}{2} - \frac{2ef}{t^2}$. After removing t from denominators, the above equality modifies to

$$\frac{-t^2 h dh - 2f de - 2edf}{de \wedge dh \wedge df} = t^2 h \partial_e \wedge \partial_f + 2e \partial_h \wedge \partial_e + 2f \partial_f \wedge \partial_h.$$

In particular, for $\tilde{\mathfrak{q}}$, we have $dF^\bullet/\omega = \tilde{\pi}$.

Example 3.7 illustrates a general phenomenon. Let D_t be the degree in t of the determinant of the map $\varphi_t : (\mathbb{K}^n)^* \rightarrow (\mathbb{K}^n)^*$, where \mathbb{K}^n is identified with \mathbb{A}^n . In case of a linear (in t) contraction of a Lie algebra \mathfrak{q} , we have $D_t = \dim V$ and φ_t multiplies the canonical volume form ω by t^{D_t} .

Theorem 3.8. *Suppose we have a contraction $\pi_t \rightsquigarrow \tilde{\pi}$ of a Poisson structure π on $\mathbb{A}^n \cong \mathbb{K}^n$ given by a family of linear automorphisms $\varphi_t : (\mathbb{K}^n)^* \rightarrow (\mathbb{K}^n)^*$ with the determinants t^{D_t} . Suppose further that $\text{ind } \mathcal{A} = \ell$ and it stays the same under the contraction. If the Kostant equality holds for a set of polynomials $F_1, \dots, F_\ell \in Z(\mathcal{A}, \pi)$, then*

- (i) $\sum \deg_t F_i \geq D_t$, moreover, if $\sum \deg_t F_i > D_t$, then F_i^\bullet are algebraically dependent;
- (ii) if $\sum \deg_t F_i = D_t$, then F_i^\bullet are algebraically independent and satisfy the Kostant equality with $\tilde{\pi}$;
- (iii) if we have an equality in (i), $\dim \text{Sing } \tilde{\pi} \leq n - 2$, and each F_i^\bullet is a homogeneous polynomial, then F_i^\bullet generate $Z(\mathcal{A}, \tilde{\pi})$.

Proof. We are contracting, so to say, both sides in the Kostant equality. For each non-zero t , we have

$$\frac{d\varphi_t^{-1}(F_1) \wedge \dots \wedge d\varphi_t^{-1}(F_\ell)}{\varphi_t^{-1}(\omega)} = \Lambda^{(n-\ell)/2} \pi_t$$

and therefore

$$\frac{t^{D_t} d\varphi_t^{-1}(F_1) \wedge \dots \wedge d\varphi_t^{-1}(F_\ell)}{\omega} = \Lambda^{(n-\ell)/2} \tilde{\pi} + tR,$$

where $R \in W^{n-\ell}[t]$.

If $\sum \deg_t F_i < D_t$, then taking the limit at $t \rightarrow 0$, we get zero on the left hand side. Since index remains the same under this contraction, $\Lambda^{(n-\ell)/2} \tilde{\pi} \neq 0$, and this proves the inequality $\sum \deg_t F_i \geq D_t$. Further, if $\sum \deg_t F_i > D_t$, then $t^{D_t}(dF_1^\bullet \wedge \dots \wedge dF_\ell^\bullet)$ is either zero or tends to infinity as t tends to zero and therefore $dF_1^\bullet \wedge \dots \wedge dF_\ell^\bullet$ must be zero. This completes the proof of part (i).

The equality $\sum \deg_t F_i = D_t$ implies that the left hand side tends to $(dF_1^\bullet \wedge \dots \wedge dF_\ell^\bullet)/\omega$ as t tends to zero. Therefore these highest components are algebraically independent and indeed satisfy the Kostant equality with $\Lambda^{(n-\ell)/2} \tilde{\pi}$ on the right hand side.

Part (ii) implies that $\mathcal{J}(F_1^\bullet, \dots, F_\ell^\bullet)$ is equal to $\text{Sing } \tilde{\pi}$. Thus, if the conditions in (iii) are satisfied, then the Jacobian locus of F_i^\bullet has dimension at most $n - 2$. Since $\text{tr. deg } Z(\mathcal{A}, \tilde{\pi}) \leq \ell$, for each $F \in Z(\mathcal{A}, \tilde{\pi})$, the polynomials $F_1^\bullet, \dots, F_\ell^\bullet$, and F are algebraically dependent. The assumption that each F_i^\bullet is homogeneous, allows us to use a characteristic zero version of Skryabin's result, see [PPY, Theorem 1.1], which states that in this situation F lies in the subalgebra generated by F_i^\bullet . \square

4. SYMMETRIC INVARIANTS OF \mathbb{Z}_2 -CONTRACTIONS

Let G be a connected reductive algebraic group defined over \mathbb{K} . Suppose that $\mathbb{K} = \overline{\mathbb{K}}$. Set $\mathfrak{g} = \text{Lie } G$. Let σ be an involution (automorphism of order 2) of G . On the Lie algebra level σ induces a \mathbb{Z}_2 -grading $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{g}^\sigma = \text{Lie } G_0$ and $G_0 := G^\sigma$ is the subgroup of σ -invariant points. In this context, G_0 is said to be a *symmetric subgroup*, G/G_0 a *symmetric space* and $(\mathfrak{g}, \mathfrak{g}_0)$ a *symmetric pair*. One can contract \mathfrak{g} to a semidirect product $\tilde{\mathfrak{g}} = \mathfrak{g}_0 \ltimes \mathfrak{g}_1$, where \mathfrak{g}_1 becomes an Abelian ideal, in the same way as described in

Section 3. We will call the resulting Lie algebra, $\tilde{\mathfrak{g}}$, a \mathbb{Z}_2 -contraction of \mathfrak{g} . In this section, our main objects of interest are \mathbb{Z}_2 -contractions of simple (non-Abelian) Lie algebras.

Set $\ell = \text{ind } \mathfrak{g} = \text{rk } \mathfrak{g}$. By [P07, Proposition 2.5], $\text{ind } \tilde{\mathfrak{g}} = \ell$ for a \mathbb{Z}_2 -contraction of a reductive Lie algebra. It was also conjectured in [P07] that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is a polynomial algebra in ℓ variables. It would be sufficient to prove the conjecture for symmetric pairs with simple \mathfrak{g} . For many pairs it was already proved in [P07]. Here we consider 4 of the remaining ones. This does not cover all them and does not prove Panyushev's conjecture.

Proposition 4.1 ([P07, Section 6]). *Suppose that \mathfrak{g} is a simple non-Abelian Lie algebra. Then all symmetric pairs $(\mathfrak{g}, \mathfrak{g}_0)$ such that the polynomiality of $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is not established yet are listed below.*

Exceptional Lie algebras:

- (E_6, F_4) , $(E_7, E_6 \oplus \mathbb{K})$, $(E_8, E_7 \oplus \mathfrak{sl}_2)$, and $(E_6, \mathfrak{so}_{10} \oplus \mathfrak{so}_2)$;
- $(E_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$.

Classical Lie algebras:

- $(\mathfrak{sp}_{2n+2m}, \mathfrak{sp}_{2n} \oplus \mathfrak{sp}_{2m})$ for $n \geq m$;
- $(\mathfrak{so}_{2\ell}, \mathfrak{gl}_\ell)$;
- $(\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n})$.

The first 4 exceptional symmetric pairs are collected in one item, because there are no good generating systems in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with respect to the corresponding \mathbb{Z}_2 -contractions, see [P07, Remark 4.3]. Moreover, it is quite possible that the algebra of symmetric invariants is not freely generated for these $\tilde{\mathfrak{g}}$. These are precisely the symmetric pairs such that the restriction homomorphism $\mathbb{K}[\mathfrak{g}]^G \rightarrow \mathbb{K}[\mathfrak{g}_1]^{G_0}$ is not surjective [H].

According to [P07, Theorem 3.3.], the Lie algebra $\tilde{\mathfrak{g}}$ always possesses the “codim-2” property, $\dim \text{Sing } \tilde{\pi} \leq \dim \tilde{\mathfrak{g}} - 2$. For the pair $(E_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$ and the three classical series listed in Proposition 4.1, we will construct homogeneous generators $F_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\sum \deg_t F_i \leq \dim \mathfrak{g}_1$ and using Theorem 3.8 prove that Panyushev's conjecture holds for them.

For each element $x \in \mathfrak{g}_1$, we let $\mathfrak{g}_{i,x} = \mathfrak{g}_x \cap \mathfrak{g}_i$ denote its centraliser in \mathfrak{g}_i ($i = 0, 1$). Let $\mathfrak{c} \subset \mathfrak{g}_1$ be a maximal (Abelian) subalgebra consisting of semisimple elements. Any such subalgebra is called a *Cartan subspace* of \mathfrak{g}_1 . Let $\mathfrak{l} = \mathfrak{g}_{0,\mathfrak{c}}$ be the centraliser of \mathfrak{c} in \mathfrak{g}_0 . We will need a few facts that can be found in e.g. [KR, Thm. 1&Prop. 8]. First, all Cartan subspaces are G_0 -conjugate. Second, $\mathfrak{l} = \mathfrak{g}_{0,s}$ for a generic $s \in \mathfrak{c}$ and therefore it is a reductive subalgebra. And finally, $G_0\mathfrak{c}$ is a dense subset of \mathfrak{g}_1 .

Let $L := (G_{0,\mathfrak{c}})^\circ$ be the connected component of the identity of $G_{0,\mathfrak{c}} = \{g \in G_0 \mid gs = s \text{ for all } s \in \mathfrak{c}\}$. Using the Killing form, we identify $\mathfrak{g} \cong \mathfrak{g}^*$, $\mathfrak{g}_1 \cong \mathfrak{g}_1^*$, and $\mathfrak{g}_0 \cong \mathfrak{g}_0^*$. Fix also the dual decomposition $\tilde{\mathfrak{g}}^* = \mathfrak{g}_0^* \oplus \mathfrak{g}_1^*$. In order to avoid confusion, let $\hat{\mathfrak{l}}$ and $\hat{\mathfrak{c}}$ denote the subspaces of $\tilde{\mathfrak{g}}^*$ arising from \mathfrak{l} and \mathfrak{c} , respectively, under this identification. The orthogonal complements appearing below are taken with respect to the Killing form of \mathfrak{g} . Let

$\tilde{G} = G_0 \ltimes \exp(\mathfrak{g}_1)$ be an algebraic group with $\text{Lie } \tilde{G} = \tilde{\mathfrak{g}}$. The group G_0 is not necessary connected and therefore \tilde{G} can also have several connected components. However, note that each bi-homogeneous with respect to the decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ component of $H \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ is an invariant of G_0 and therefore in view of Lemma 3.1, $H^{\bullet} \in \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{G}}$.

Remark 4.2. In [P07], a good generating system (g.g.s.) consists of homogeneous polynomials by the definition. It is possible to show that if there is a g.g.s. in $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with respect to a contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$, then there are also homogeneous polynomials forming a g.g.s.. We will not use and therefore will not prove this fact. In this and the following sections, all generating systems of invariants contain only homogeneous polynomials.

Example 4.3. Take $(\mathfrak{g}, \mathfrak{g}_0) = (E_7, \mathfrak{so}_{12} \oplus \mathfrak{sl}_2)$. Then $D_t = \dim \mathfrak{g}_1 = 64$, and the generating homogeneous invariants $H_1, \dots, H_7 \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ have degrees: 2, 6, 8, 10, 12, 14, 18. It is known that the restrictions of H_1, H_2, H_3 , and H_5 to \mathfrak{g}_1^* generate $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}_0}$, independently of the choice of H_i , see [H]. We will show that there is a g.g.s. in $Z\mathcal{S}(\mathfrak{g})$ with respect to the contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$.

Take any of the remaining three generators, say H_j . Assume that $H_j^{\bullet} \in \mathcal{S}(\mathfrak{g}_1)$. Then it can be expressed as a polynomial P in H_i^{\bullet} with $i \in \{1, 2, 3, 5\}$ and we can replace H_j by $H_j - P(H_1, H_2, H_3, H_5)$. Or rather assume from the beginning that $\deg_t H_j < \deg H_j$.

Next step is to show that $\deg_t H_j < \deg H_j - 1$. Assume this not to be the case. Restricting H_j^{\bullet} to $\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}}$, we get either zero or an L -invariant polynomial function of bi-degree $(\deg H_j - 1, 1)$, in other words, a sum of L -invariants in $\mathcal{S}(\mathfrak{l})$ of an odd degree with coefficients from \mathfrak{c} . In this example $\mathfrak{l} = \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2$ (see e.g. [VO, §5.4 and Table 9 in Ref. Chapter]) and all symmetric invariants have even degrees. This shows that H_j^{\bullet} is zero on $\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}}$. Clearly H_j^{\bullet} is also zero on the \tilde{G} -saturation $\tilde{G}(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})$.

Consider first the action of $\exp(\mathfrak{g}_1) \subset \tilde{G}$. Note that $[\mathfrak{g}, x] = \mathfrak{g}_x^{\perp}$ for any $x \in \mathfrak{g}$, and hence $[\mathfrak{g}_1, x] = \mathfrak{g}_0 \cap \mathfrak{g}_x^{\perp} = \mathfrak{g}_0 \cap (\mathfrak{g}_{0,x})^{\perp}$ for any $x \in \mathfrak{g}_1$. Now let $\hat{s} \in \hat{\mathfrak{c}}$ be an element coming from some $s \in \mathfrak{c}$. Then

$$\exp(\mathfrak{g}_1)(\hat{\mathfrak{l}} \times \{\hat{s}\}) = \hat{\mathfrak{l}} \times \{\hat{s}\} + \widehat{[\mathfrak{g}_1, s]},$$

where $\widehat{[\mathfrak{g}_1, s]}$ is the annihilator of $\mathfrak{g}_{0,s}$ in \mathfrak{g}_0^* . Since $\mathfrak{g}_{0,s} = \mathfrak{l}$ for generic $s \in \mathfrak{c}$, the saturation $\exp(\mathfrak{g}_1)(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})$ is a dense subset of $\mathfrak{g}_0^* \oplus \hat{\mathfrak{c}}$. Applying G_0 to this subset, we get that $\tilde{G}(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}}) = \tilde{\mathfrak{g}}^*$ and therefore $H_j^{\bullet} = 0$. Since the highest t -component of a non-zero polynomial is non-zero, we get that $\deg_t H_j \leq \deg H_j - 2$. Summing up,

$$\sum_{i=1}^7 \deg_t H_i \leq \sum \deg H_i - 6 = 70 - 6 = 64 = D_t.$$

Multiplying one of the H_i by a non-zero constant, we may assume that H_1, \dots, H_{ℓ} satisfy the Kostant equality. Then, by Theorem 3.8(i),(ii), H_i^{\bullet} are algebraically independent, which means that H_i form a good generating system.

In order to simplify calculations for other pairs, we prove a simple equality concerning ranks and dimensions. Recall that $\ell = \text{rk } \mathfrak{g}$.

Lemma 4.4. *Let $\mathfrak{b}_\mathfrak{l} \subset \mathfrak{l}$ be a Borel subalgebra. Then $\dim \mathfrak{b} = \dim \mathfrak{g}_1 + \dim \mathfrak{b}_\mathfrak{l}$.*

Proof. Clearly the subspace $\mathfrak{l} \oplus \mathfrak{c}$ contains a maximal torus of \mathfrak{g} . Therefore $\ell = \text{rk } \mathfrak{l} + \dim \mathfrak{c}$. It is known that the dimension of a maximal G_0 -orbit in \mathfrak{g}_1 equals $\dim \mathfrak{g}_0 - \dim \mathfrak{l}$ on one hand, and $\dim \mathfrak{g}_1 - \dim \mathfrak{c}$ on the other, see e.g. [KR, Proposition 9]. Consequently, $\dim \mathfrak{g}_0 - \dim \mathfrak{l} = \dim \mathfrak{g}_1 - \dim \mathfrak{c}$. Thereby we have

$$\begin{aligned} \dim \mathfrak{b} &= (\dim \mathfrak{g} + \ell)/2 = (\dim \mathfrak{g}_0 + \dim \mathfrak{g}_1 + \ell)/2 = \\ &= (\dim \mathfrak{g}_1 + \dim \mathfrak{l} - \dim \mathfrak{c} + \dim \mathfrak{g}_1 + \ell)/2 = \dim \mathfrak{g}_1 + \\ &\quad + (\dim \mathfrak{l} - \dim \mathfrak{c} + \text{rk } \mathfrak{l} + \dim \mathfrak{c})/2 = \dim \mathfrak{g}_1 + \dim \mathfrak{b}_\mathfrak{l}. \end{aligned}$$

□

The following assertion was predicted by D. Panyushev.

Theorem 4.5. *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a symmetric pair with \mathfrak{g} simple. Suppose that the restriction map $\mathbb{K}[\mathfrak{g}]^G \rightarrow \mathbb{K}[\mathfrak{g}_1]^{G_0}$ is surjective. Then there is a good generating system F_1, \dots, F_ℓ in $\mathcal{S}(\mathfrak{g})^\mathfrak{g}$ such that each F_i is homogeneous and $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by F_i^\bullet .*

Proof. According to [P07], there are 4 pairs to consider. For one of them the existence of a g.g.s. was established in Example 4.3. Our next goal is to construct good generating systems for 3 classical pairs listed in Proposition 4.1. We always assume that a set of generators F_1, \dots, F_ℓ in $\mathcal{S}(\mathfrak{g})^\mathfrak{g}$ is normalised in order to satisfy the Kostant equality.

For the first pair, with $\mathfrak{g} = \mathfrak{sp}_{2n+2m}$, we start with a set of generating invariants $\{H_1, \dots, H_\ell\} \subset \mathcal{S}(\mathfrak{g})^\mathfrak{g}$, where each H_i is the sum of all principal $2i$ -minors (this is also a coefficient of the characteristic polynomial). As can be readily seen from the block structure of this symmetric pair (Figure 1), $\deg_t H_i \leq 4m$ for all i . To be more precise, for $1 \leq i \leq m$,

\mathfrak{sp}_{2n}	\mathfrak{g}_1
\mathfrak{g}_1	\mathfrak{sp}_{2m}

Fig. 1. Symmetric decomposition of \mathfrak{sp}_{2n+2m}

the highest t -components H_i^\bullet lie in $\mathcal{S}(\mathfrak{g}_1)$ and form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$. Therefore set $F_i := H_i$ for $i \leq m$. Here $\mathfrak{l} = (\mathfrak{sl}_2)^m \oplus \mathfrak{sp}_{2n-2m}$ and all symmetric \mathfrak{l} -invariants are of even degrees. Applying the same trick as in Example 4.3, we can modify H_j with $m < j \leq 2m$

to F_j in such a way that $\deg_t F_j \leq 2j - 2$. Remaining generators stay as they are, $F_i = H_i$ for $i > 2m$. Summing up

$$\sum_{i=1}^{\ell} (\deg F_i - \deg_t F_i) \geq 2m + \sum_{j=1}^{n-m} 2j = \dim \mathfrak{b}_l.$$

Making use of Lemma 4.4 and again of the equality $\sum \deg F_i = \dim \mathfrak{b}$, we get that $\sum \deg_t F_i \leq \dim \mathfrak{g}_1 = D_t$.

For the second pair, with $\mathfrak{g} = \mathfrak{so}_{2\ell}$, we have $\mathfrak{l} = (\mathfrak{sl}_2)^{\ell/2}$, then ℓ is even, and $\mathfrak{l} = (\mathfrak{sl}_2)^{[\ell/2]} \oplus \mathfrak{so}_2$, then ℓ is odd. In case ℓ is even, we argue as in Example 4.3. Choose homogeneous generators $F_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that the highest components $F_1^{\bullet}, \dots, F_{\ell/2}^{\bullet}$ form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$, and $\deg_t F_i \leq \deg F_i - 2$ for $i > \ell/2$. Then, taking into account Lemma 4.4, we get

$$\sum_{i=1}^{\ell} \deg_t F_i \leq \sum_{i=1}^{\ell} \deg F_i - \ell = \dim \mathfrak{b} - \dim \mathfrak{b}_l = \dim \mathfrak{g}_1 = D_t.$$

The case of odd ℓ is more interesting. We begin with a set of generating invariants $\{H_1, \dots, H_{\ell}\} \subset \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$, where each H_i with $i \neq \ell$ is the sum of all principal $2i$ -minors and H_{ℓ} is the pfaffian, in particular, $\deg F_{\ell} = \ell$ is odd. One can realise $\mathfrak{so}_{2\ell}$ as a set of $2\ell \times 2\ell$ matrices skew-symmetric with respect to the anti-diagonal. Then elements of \mathfrak{g}_1 have block structure as shown in Figure 2. This implies that all bi-homogenous (in \mathfrak{g}_0 and \mathfrak{g}_1)

0	B
C	0

Fig. 2. \mathfrak{g}_1 for $(\mathfrak{so}_{2\ell}, \mathfrak{gl}_{\ell})$. Here the matrices B and C are skew-symmetric with respect to the anti-diagonal.

components of H_i with $i < \ell$ have even degrees in \mathfrak{g}_1 (and in \mathfrak{g}_0).

The highest t -components H_i^{\bullet} with $2i < \ell$ form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$. Therefore we put $F_i := H_i$ for these i . Each H_j with $(\ell/2) < j < \ell$ can be modified to F_j with $\deg_t F_j \leq 2j - 2$. And, finally, since $\det \xi = 0$ for all $\xi \in \mathfrak{g}_1$, we have $\deg_t H_{\ell} \leq \ell - 1$. Set $F_{\ell} := H_{\ell}$. Then

$$\sum_{i=1}^{\ell} \deg_t F_i \leq \sum_{i=1}^{\ell} \deg F_i - (\ell - 1) - 1 = \dim \mathfrak{b} - \dim \mathfrak{b}_l = \dim \mathfrak{g}_1 = D_t,$$

where again we have used Lemma 4.4.

For the third pair, with $\mathfrak{g} = \mathfrak{sl}_{2n}$, we have $\mathfrak{l} = (\mathfrak{sl}_2)^n$. Here everything works exactly as in Example 4.3. We take homogeneous invariants F_i with $\deg F_i = i + 1$. Then F_i^{\bullet} with

$1 \leq i < n$ form a generating set in $\mathbb{K}[\mathfrak{g}_1^*]^{G_0}$ and, modifying F_j if necessary, we can assume that $\deg_t F_j^\bullet \leq \deg F_j - 2$ for $j \geq n$. In view of Lemma 4.4,

$$\sum_{i=1}^{\ell} \deg_t F_i \leq \sum_{i=1}^{\ell} \deg F_i - 2n = \dim \mathfrak{b} - \dim \mathfrak{b}_t = \dim \mathfrak{g}_1 = D_t.$$

For all three series we have constructed $F_i \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ such that $\sum_{i=1}^{\ell} \deg_t F_i \leq D_t$. By Theorem 3.8(i),(ii), the polynomials F_i form a g.g.s.. Since in addition all F_i are homogeneous here as well as in Example 4.3, the polynomials F_i^\bullet generate $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ by [P07, Theorem 4.2(i)] or by Theorem 3.8(iii), if one recalls that $\tilde{\mathfrak{g}}$ has the “codim-2” property [P07, Theorem 3.3.]. \square

Corollary 4.6 (cf. [P07, Theorem 4.2(ii)]). *Let $(\mathfrak{g}, \mathfrak{g}_0)$ be a symmetric pair such that Theorem 4.5 applies. Then the Lie algebra $\tilde{\mathfrak{g}}$ is of Kostant type.*

4.1. Poisson semicentre.

Definition 4.7. Let \mathfrak{q} be a Lie algebra. Then an elements $H \in \mathcal{S}(\mathfrak{q})$ is called a *semi-invariant* if $\{\xi, H\} \in \mathbb{K}H$ for all $\xi \in \mathfrak{q}$. We let $\mathcal{S}(\mathfrak{q})_{\text{si}}$ denote the \mathbb{K} -algebra generated by semi-invariants. This algebra is also called the Poisson semicentre of $\mathcal{S}(\mathfrak{q})$.

One of the easy to deduce properties of the semi-invariants is that $\{\mathcal{S}(\mathfrak{q})_{\text{si}}, \mathcal{S}(\mathfrak{q})_{\text{si}}\} = 0$, see e.g. [OV, Section 2]. Recently Poisson semicentres were studied in [OV] and [JSh]. In particular, [OV] proves a degree inequality for Lie algebras \mathfrak{q} such that $Z\mathcal{S}(\mathfrak{q})$ is a polynomial ring and $Z\mathcal{S}(\mathfrak{q}) = \mathcal{S}(\mathfrak{q})_{\text{si}}$. Here we show that some \mathbb{Z}_2 -contractions $\tilde{\mathfrak{g}}$ of simple Lie algebras also satisfy the second property. If G_0 is semisimple, then $\tilde{\mathfrak{g}}$ has no non-trivial characters and clearly $Z\mathcal{S}(\tilde{\mathfrak{g}}) = \mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}}$.

Until the end of this section we assume that G_0 has a non-trivial connected centre. Since \mathfrak{g} is simple, the centre of G_0 is 1-dimensional, see e.g. [P07, Section 6] (on a classification free basis this fact follows from a description of the finite order automorphisms of \mathfrak{g} in terms of Kac diagrams). Let G'_0 be the derived group of G_0 and $\mathfrak{g}'_0 = [\mathfrak{g}_0, \mathfrak{g}_0]$ the derived Lie algebra. By an elementary observation that $\mathfrak{g}_0 \oplus [\mathfrak{g}_0, \mathfrak{g}_1]$ is an ideal of \mathfrak{g} , one proves the equality $\tilde{\mathfrak{g}}' = \mathfrak{g}'_0 \ltimes \mathfrak{g}_1$.

Recall that a symmetric space (or a symmetric pair) can be either of tube type, meaning $\mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}'_0} \neq \mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0}$, or non-tube type. For example, $(\mathfrak{so}_{2\ell}, \mathfrak{gl}_\ell)$ is of tube type if and only if ℓ is even. There are many characterisations of symmetric pairs of tube type. If $(\mathfrak{g}, \mathfrak{g}_0)$ is of tube type, then there are more semi-invariants than symmetric $\tilde{\mathfrak{g}}$ -invariants, because $\mathcal{S}(\mathfrak{g}_1)^{\mathfrak{g}'_0} \subset \mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}}$. That case may be worth of investigating. Here we deal with symmetric spaces of non-tube type.

The following observation helps to treat semi-direct products (cf. [R] or [P07', Proposition 5.5]).

Lemma 4.8. *Let $\mathfrak{q} = \mathfrak{f} \ltimes V$ be a semi-direct product of a Lie algebra \mathfrak{f} and an Abelian ideal V . Take $x = a + b \in \mathfrak{q}^*$ with $a(V) = 0 = b(\mathfrak{f})$. Let $\mathfrak{a} = \text{Ann}(\mathfrak{f} \cdot b)$ be a subspace of $(V^*)^* = V$. Then $\mathfrak{a} \triangleleft \mathfrak{q}_x$ and $\mathfrak{q}_x/\mathfrak{a} \cong (\mathfrak{f}_b)_{\tilde{a}}$, where \tilde{a} is the restriction of a to \mathfrak{f}_b .*

Proof. Note that $V \cdot b$ is zero on V , because $[V, V] = 0$, and therefore $\mathfrak{q}_x \subset \mathfrak{f}_b \ltimes V$. It is also quite clear that $V \cdot b \subset \text{Ann}(\mathfrak{f}_b \oplus V)$. Hence $\mathfrak{q}_x \subset (\mathfrak{f}_b)_{\tilde{a}} \ltimes V$. By the dimension reasons, $V \cdot b = \text{Ann}(\mathfrak{f}_b \oplus V)$ and for each $\xi \in (\mathfrak{f}_b)_{\tilde{a}}$, there is $\eta \in V$ such that $\eta \cdot b = \xi \cdot a$. It remains to notice that $\mathfrak{q}_x \cap V = \text{Ann}(\mathfrak{f} \cdot b)$. \square

Proposition 4.9. *Suppose that $(\mathfrak{g}, \mathfrak{g}_0)$ is a symmetric pair of non-tube type. Then $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} = Z\mathcal{S}(\tilde{\mathfrak{g}})$.*

Proof. Here we consider the connected groups \tilde{G}° and $(\tilde{G}')^\circ = (G'_0)^\circ \ltimes \exp(\mathfrak{g}_1)$. Each character of \tilde{G}° is trivial on $(\tilde{G}')^\circ$, hence $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} \subset \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}'}}$. (In fact, $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} = \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}'}}$). Next we take $H \in \mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}'}}$ and show that it is an invariant of \tilde{G}° .

Since \mathfrak{g}'_0 is semisimple, $\mathbb{K}(\mathfrak{g}_1)^{\mathfrak{g}'_0}$ is the quotient field of $\mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}'_0}$. By Rosenlicht's theorem, generic orbits of an algebraic group, in our case $(G'_0)^\circ$, are separated by rational invariants. Thereby the equality $\mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}'_0} = \mathbb{K}[\mathfrak{g}_1]^{\mathfrak{g}_0}$ implies that G_0° and $(G'_0)^\circ$ have the same generic orbits in \mathfrak{g}_1 and \mathfrak{g}_1^* . On the Lie algebra level this means that $\mathfrak{g}_0 = \mathfrak{g}'_0 + \mathfrak{g}_{0,b}$ for generic $b \in \mathfrak{g}_1$.

Suppose that $x = \hat{a} + \hat{b} \in \tilde{\mathfrak{g}}^*$, where \hat{a} and \hat{b} correspond to generic $a \in \mathfrak{l}$ and $b \in \mathfrak{c}$, respectively. In view of Lemma 4.8 and the fact that $\mathfrak{l} = \mathfrak{g}_{0,b}$ is reductive, $\tilde{\mathfrak{g}}_x = \mathfrak{l}_a \ltimes ([\mathfrak{g}_0, b])^\perp$, where $([\mathfrak{g}_0, b])^\perp = \{\eta \in \mathfrak{g}_1 \mid [b, \eta] \in \mathfrak{g}_0^\perp\}$ (the orthogonal complement is taken with respect to the Killing form of \mathfrak{g}).

Since $\mathfrak{g}_0 = \mathfrak{g}'_0 + \mathfrak{l}$ and \mathfrak{l}_a contains the centre of \mathfrak{l} , we have also $\mathfrak{g}_0 = \mathfrak{g}'_0 + \mathfrak{l}_a$ and $\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}' + \tilde{\mathfrak{g}}_x$. This leads to the equalities $\tilde{\mathfrak{g}}' \cdot x = \tilde{\mathfrak{g}} \cdot x$ and $\dim \tilde{G}'x = \dim \tilde{G}x$. In addition, $(\tilde{G}')^\circ$ is a normal subgroup of \tilde{G}° . Consequently, $(\tilde{G}')^\circ x = \tilde{G}^\circ x$. This equality holds on an open subset of $\tilde{G}^\circ(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}})$, which is a dense subset of $\tilde{\mathfrak{g}}^*$, because $\tilde{G}(\hat{\mathfrak{l}} \oplus \hat{\mathfrak{c}}) = \tilde{\mathfrak{g}}^*$, as we know from Example 4.3, and $\tilde{\mathfrak{g}}^*$ is irreducible. Thus, H is constant on a generic \tilde{G}° -orbit and hence $H \in Z\mathcal{S}(\tilde{\mathfrak{g}})$. \square

5. APPLICATIONS TO E. FEIGIN'S CONTRACTION

In this section, $\mathfrak{g} = \text{Lie } G$ is a simple Lie algebra of rank ℓ , $B \subset G$ is a Borel subgroup, and $\mathfrak{b} = \text{Lie } B$ is a Borel subalgebra. We keep the assumption that $\mathbb{K} = \overline{\mathbb{K}}$. Fix a decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where \mathfrak{n}^- is the nilpotent radical of an opposite Borel, and consider a one-parameter contraction of \mathfrak{g} given by this decomposition. For the resulting Lie algebra $\tilde{\mathfrak{g}}$, we have $\tilde{\mathfrak{g}} = \mathfrak{b} \ltimes \mathfrak{n}^-$, where \mathfrak{n}^- is an Abelian ideal. This contraction was recently introduced by E. Feigin in [F10]. His motivation came from some problems in representation theory [FFL]. Degenerations of flag varieties of \mathfrak{g} related to the contraction $\mathfrak{g} \rightsquigarrow \tilde{\mathfrak{g}}$ were further studied in [F11] and [FFiL].

Let $\{\alpha_1, \dots, \alpha_\ell\}$ be a set of the simple roots and e_i, f_i corresponding elements of the Chevalley basis. Set $\mathfrak{g}_{\text{reg}} = \{x \in \mathfrak{g} \mid \dim \mathfrak{g}_x = \ell\}$, where \mathfrak{g}_x is the stabiliser in the adjoint representation, $\mathfrak{n}_{\text{reg}} := \mathfrak{n} \cap \mathfrak{g}_{\text{reg}}$. If $x \in \mathfrak{n}_{\text{reg}}$, then $\mathfrak{n}_x = \mathfrak{b}_x = \mathfrak{g}_x$ and Bx is a dense open orbit in \mathfrak{n} . Hence $\mathfrak{n}_{\text{reg}}$ is a single B -orbit. The complement of this orbit was described

by Kostant [K, Theorem 4] and $\mathfrak{n} \setminus \mathfrak{n}_{\text{reg}} = \bigcup_{i=1}^{\ell} \mathfrak{D}_i$, where each \mathfrak{D}_i is a linear subspace of dimension $\dim \mathfrak{n} - 1$ orthogonal to f_i . We will also need an interpretation of \mathfrak{D}_i as closures of orbital varieties. It is a classical fact that for each nilpotent orbit $Ge \subset \mathfrak{g}$, all irreducible components of $Ge \cap \mathfrak{n}$ are of dimension $\frac{1}{2} \dim Ge$. In particular, $\mathfrak{n} \setminus \mathfrak{n}_{\text{reg}} = \overline{\mathcal{O}^{\text{sub}} \cap \mathfrak{n}}$, where \mathcal{O}^{sub} is the unique nilpotent G -orbit in \mathfrak{g} of dimension $\dim \mathfrak{g} - \ell - 2$.

Making further use of the Killing form $(\ , \)$ of \mathfrak{g} , we identify $(\mathfrak{n}^-)^*$ with the nilpotent radical $\mathfrak{n} \subset \mathfrak{b}$ and fix the dual decomposition $\tilde{\mathfrak{g}}^* = \mathfrak{b}^* \oplus \mathfrak{n}^{\text{ab}}$, where \mathfrak{n}^{ab} indicates that \mathfrak{n}^{ab} is a space of the linear functions on an Abelian ideal. Let also $\mathfrak{n}_{\text{reg}}^{\text{ab}}$ be a subset of $\mathfrak{n}^{\text{ab}} \subset \tilde{\mathfrak{g}}^*$ corresponding to $\mathfrak{n}_{\text{reg}}$. We identify \mathfrak{D}_i with subsets of \mathfrak{n}^{ab} using the same letters for them.

Next statement was first proved in [PY].

Lemma 5.1 (cf. Lemma 4.8). *We have $\text{ind } \tilde{\mathfrak{g}} = \ell$.*

Proof. Clearly, $\text{rk } \pi$ cannot get larger after a contraction, therefore $\text{ind } \tilde{\mathfrak{g}} \geq \ell$. On the other hand, take $x \in \mathfrak{n}_{\text{reg}}^{\text{ab}}$ and extend it to a linear function on $\tilde{\mathfrak{g}}^*$ by putting $x(\mathfrak{b}) = 0$. Then $\tilde{\mathfrak{g}}_x = \mathfrak{b}_x = \mathfrak{n}_x$ and it has dimension ℓ . Thus $\text{ind } \tilde{\mathfrak{g}} = \ell$. \square

Another result of [PY], Theorem 3.3, states that $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$ is freely generated by some polynomials \hat{P}_i (with $1 \leq i \leq \ell$). The construction of these polynomials \hat{P}_i starts with a system of homogeneous generators F_i of $\mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ with $\deg F_i \leq \deg F_{i+1}$. It is also shown that $\hat{P}_i = F_i^\bullet$, [PY, Theorem 3.9]. We assume that F_i are normalised to satisfy the Kostant equality.

Lemma 5.2. *Let F_i be as above. Then F_i^\bullet satisfy the Kostant equality with $\tilde{\pi}$ and therefore $\tilde{\mathfrak{g}}$ is a Lie algebra of Kostant type. Besides, $\deg_t F_i = \deg F_i - 1$ for all i .*

Proof. Recall that $\mathcal{S}(\mathfrak{n}^-)^{\mathfrak{b}} = \mathbb{K}$. If $\deg_t F_i = \deg F_i$, i.e., $F_i^\bullet \in \mathcal{S}(\mathfrak{n}^-)$, then also $F_i^\bullet \in \mathcal{S}(\mathfrak{n}^-)^{\mathfrak{b}}$. A contradiction. Hence $\deg_t F_i \leq \deg F_i - 1$ for each i and

$$\sum \deg_t F_i \leq \dim \mathfrak{b} - \ell = \dim \mathfrak{n} = D_t.$$

By Theorem 3.8(i),(ii), $\deg_t F_i = \deg F_i - 1$, the polynomials F_i^\bullet are algebraically independent and satisfy the Kostant equality with $\tilde{\pi}$. Since, according to [PY, Section 3], F_i^\bullet generate $\mathcal{S}(\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}$, the Lie algebra $\tilde{\mathfrak{g}}$ is of Kostant type. \square

Actually, the bi-degrees of F_i^\bullet with respect to the decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$ have been already found in [PY].

Remark 5.3. Lemma 5.2 implies that $\mathcal{J}(F_1^\bullet, \dots, F_\ell^\bullet) = \text{Sing } \tilde{\pi}$. Therefore $\text{Sing } \tilde{\pi}$ contains a divisor whenever \mathfrak{g} is not of type A , see [PY, Th. 4.2&Prop. 4.3]. This means that outside of type A we get curious examples of Lie algebras of Kostant type that does not have the “codim-2” property.

Next we turn our attention to the subset $\text{Sing } \tilde{\pi} = \tilde{\mathfrak{g}}_{\text{sing}}^*$. The proof of Lemma 5.1 shows that $\mathfrak{b}^* \times \mathfrak{n}_{\text{reg}}^{\text{ab}} \subset \tilde{\mathfrak{g}}_{\text{reg}}^*$ and therefore $\tilde{\mathfrak{g}}_{\text{sing}}^* \subset \mathfrak{b}^* \times \left(\bigcup_{i=1}^{\ell} \mathfrak{D}_i \right)$, where the subspaces \mathfrak{D}_i are regarded as subsets of \mathfrak{n}^{ab} .

Definition 5.4. Let \mathfrak{q} be an n -dimensional Lie algebra with $\text{ind } \mathfrak{q} = \ell$ and π its Lie-Poisson tensor. Then we will say that a polynomial p is a *fundamental semi-invariant* of \mathfrak{q} , if $\Lambda^{(n-\ell)/2} \pi = pR$ with $R \in W^{n-\ell}$ (notation as in Section 2) and the zero set of R in \mathfrak{q}^* has codimension greater than or equal to 2.

In [OV], the fundamental semi-invariant is defined as the greatest common divisor of the $\text{rk } \pi \times \text{rk } \pi$ (here $\text{rk } \pi = n - \ell$) minors in the matrix of π . Our polynomial is a square root of that one (up to a non-zero scalar) and is a scalar multiple of the fundamental semi-invariant in the sense of [JSh, Section 4.1].

Let δ be the highest root, e_δ a highest root vector, and $r_i = [\delta : \alpha_i]$ the i -th coefficient in the decomposition of δ , i.e., $\delta = \sum r_i \alpha_i$.

As is well-known, the highest degree of a homogeneous generator, under our assumptions, $\deg F_\ell$, equals $1 + \sum r_i$. Since F_ℓ^\bullet has weight zero and is of degree 1 in \mathfrak{b} and $(\deg F_\ell - 1)$ in \mathfrak{n}^- , up to a scalar multiple $F_\ell^\bullet = e_\delta \prod f_i^{r_i}$. This is also proved in [PY, Lemma 4.1].

Set $p := \prod f_i^{r_i-1}$. Note that in type A we have $r_i = 1$ for all i and hence $p = 1$. Here we generalise a result of [PY], Proposition 4.3, stating that in type A the singular set $\text{Sing } \tilde{\pi}$ contains no divisors.

Theorem 5.5. Let $\tilde{\mathfrak{g}}$ be Feigin's contraction of a simple Lie algebra \mathfrak{g} . Then $p = \prod f_i^{r_i-1}$ is a fundamental semi-invariant of $\tilde{\mathfrak{g}}$.

Proof. Set $\mathcal{F} = dF_1^\bullet \wedge \dots \wedge dF_\ell^\bullet$. Consider also a differential 1-form

$$L = \left(\prod_{i=1}^{\ell} f_i \right) de_\delta + e_\delta \sum_{i=1}^{\ell} r_i f_1 \dots f_{i-1} f_{i+1} \dots f_\ell df_i$$

and set $R := dF_1^\bullet \wedge \dots \wedge dF_{\ell-1}^\bullet \wedge L$. Note that $dF_\ell^\bullet = apL$ with $a \in \mathbb{K}^\times$ and therefore $\mathcal{F} = apR$. In view of the Kostant equality for F_i^\bullet established in Lemma 5.2, we have to show that the zero set of R contains no divisors.

Clearly, the zero set of R is contained in $\tilde{\mathfrak{g}}_{\text{sing}}^*$ and we have to prove that R is non-zero on each irreducible divisor in $\tilde{\mathfrak{g}}_{\text{sing}}^*$. As was already mentioned, the proof of Lemma 5.1

together with a result of Kostant [K, Theorem 4] imply that $\tilde{\mathfrak{g}}_{\text{sing}}^* \subset \mathfrak{b}^* \times \left(\bigcup_{i=1}^{\ell} \mathfrak{D}_i \right)$, where \mathfrak{D}_i

are the components of $\overline{\mathcal{O}^{\text{sub}} \cap \mathfrak{n}}$. Fix $i \in \{1, \dots, \ell\}$. There is an element $e = e(i)$ in \mathfrak{D}_i such that $e \in \mathcal{O}^{\text{sub}}$ and $(f_j, e) \neq 0$ for all $j \neq i$. Take this e and add to it $b \in \mathfrak{b}^*$ such that $b(e_\delta) \neq 0$ forming a linear function $x = b + e$ on $\tilde{\mathfrak{g}}$. Evaluating L at x we get $L_x = a' df_i$ with $a' \in \mathbb{K}^\times$.

The goal is to prove that $d_x F_j^\bullet$ with $1 \leq j < \ell$ and L_x are linear independent. To this end we calculate $d_x F_j^\bullet$.

Each $d_x F_j^\bullet$ can be considered as an element of $\tilde{\mathfrak{g}}$ and therefore $d_x F_j^\bullet = \xi_j + \eta_j$ with $\xi_j \in \mathfrak{b}$ and $\eta_j \in \mathfrak{n}^-$. Since each F_j^\bullet has degree 1 in \mathfrak{b} , $\xi_i = d_e F_i$, where e is regarded as an element of $\mathfrak{g} \cong \mathfrak{g}^*$. By e.g. [SI, Sect. 8.3, Lemma 1] or [P07', Theorem 10.6], $d_e F_j$ with $j \leq \ell$ generate a subspace of dimension $\ell - 1$. For $d_x F_\ell^\bullet$ we have two possibilities, either it is zero (if $r_i > 1$), or proportional to df_i . In any case, $\xi_\ell = 0$. Therefore $\xi_1, \dots, \xi_{\ell-1}$ are linear independent and clearly $d_x F_j^\bullet$ with $1 \leq j < \ell$ and L_x together generate a subspace of dimension ℓ . This proves that R is non-zero on each $\mathfrak{b}^* \times \mathfrak{D}_i$. Therefore $\dim\{\xi \in \tilde{\mathfrak{g}}^* \mid R_\xi = 0\} \leq \dim \tilde{\mathfrak{g}} - 2$ and we are done. \square

5.1. Proper semi-invariants. A semi-invariant is said to be *proper* if it is not an invariant. The Lie algebra $\tilde{\mathfrak{g}}$ possesses proper symmetric semi-invariants, for example e_δ . Therefore describing $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}}$ is an interesting task.

Set $H_i = F_i^\bullet$ for $1 \leq i < \ell$; $H_i = f_j$, where $j = i - \ell + 1$ for $\ell \leq i < 2\ell$; and $H_{2\ell} = e_\delta$. Clearly all these functions are semi-invariants of $\tilde{\mathfrak{g}}$. We will show that they generate $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}}$.

Let $\mathfrak{h} \subset \mathfrak{b}$ be a Cartan subalgebra of \mathfrak{g} , $U \subset B$ the unipotent radical, and $\tilde{\mathfrak{g}}'$ the derived algebra of $\tilde{\mathfrak{g}}$. We have $\tilde{\mathfrak{g}}' = \mathfrak{n} \ltimes \mathfrak{n}^-$. Note that the Lie algebra $\tilde{\mathfrak{g}}'$ has only trivial characters.

Lemma 5.6. *We have $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} = Z\mathcal{S}(\tilde{\mathfrak{g}}')$. In particular, $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}}$ is a subalgebra of $\mathcal{S}(\tilde{\mathfrak{g}}')$.*

Proof. Suppose that $\mathcal{S}(\tilde{\mathfrak{g}})_\lambda$ is an eigenspace of $\tilde{\mathfrak{g}}$ corresponding to a character $\lambda \in \tilde{\mathfrak{g}}^*$ and $\mathcal{S}(\tilde{\mathfrak{g}})_\lambda \neq 0$. Let $\tilde{\mathfrak{g}}^\lambda \subset \tilde{\mathfrak{g}}$ be the kernel of λ . Then, by a result of Borho [BGR, Satz 6.1], $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} \subset \mathcal{S}(\tilde{\mathfrak{g}}^\lambda)$ (see also [RV, Lemme 4.1] or [JSh, Sect. 1.2]). Since $f_1, \dots, f_\ell \in \mathfrak{n}^- \subset \tilde{\mathfrak{g}}$ are semi-invariants of the weights $-\alpha_i$ ($1 \leq i \leq \ell$), we conclude that $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} \subset \mathcal{S}(\tilde{\mathfrak{g}}')$. Next, $\tilde{\mathfrak{g}}'$ has no non-trivial characters and the action of \mathfrak{h} on $Z\mathcal{S}(\tilde{\mathfrak{g}}')$ is diagonalisable. Thus indeed $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}} = Z\mathcal{S}(\tilde{\mathfrak{g}}')$. \square

Lemma 5.6 shows that $\tilde{\mathfrak{g}}'$ is the canonical truncation of $\tilde{\mathfrak{g}}$ in the following sense. For any algebraic finite-dimensional Lie algebra \mathfrak{q} , there exists a unique subalgebra \mathfrak{a} such that $\mathcal{S}(\mathfrak{q})_{\text{si}} = Z\mathcal{S}(\mathfrak{a})$ [BGR]. This \mathfrak{a} is said to be the truncation of \mathfrak{q} .

Lemma 5.7. *Let H_i be as above. Then the polynomials H_i are algebraically independent. Besides, $\text{ind } \tilde{\mathfrak{g}}' = 2\ell$.*

Proof. Let $\hat{e} \in \mathfrak{n}^{\text{ab}}$ be a linear function coming from a regular nilpotent element $e \in \mathfrak{n}$. We extend it to a function on $\tilde{\mathfrak{g}}^*$ by setting $\hat{e}(\mathfrak{b}) = 0$. Then $d_{\hat{e}} F_i^\bullet = d_e F_i$. Moreover, $d_e F_\ell = e_\delta$ up to a constant. Therefore $d_e F_1, \dots, d_e F_{\ell-1}$ and $e_\delta = d_e F_\ell$ generate \mathfrak{n}_e , a subspace of dimension ℓ . The other polynomials H_j ($\ell \leq j < 2\ell$) are linear independent elements of \mathfrak{n}^- . Thus $d_{\hat{e}} H_i$ are linear independent and the first statement is proved.

According to the index formula of Raïs [R] (cf. Lemma 4.8),

$$\text{ind } \tilde{\mathfrak{g}}' = (\dim \mathfrak{n}^{\text{ab}} - \dim U\hat{e}) + \text{ind } \mathfrak{n}_e = 2\ell.$$

Alternatively, one can use [OV, Lemma 3.7.], which calculates the index of a truncated Lie algebra. In our case it reads $\text{ind } \tilde{\mathfrak{g}}' = \text{ind } \tilde{\mathfrak{g}} + (\dim \tilde{\mathfrak{g}} - \dim \tilde{\mathfrak{g}}')$. \square

Theorem 5.8. *Let $\tilde{\mathfrak{g}}$ be Feigin's contraction of \mathfrak{g} . Then $\mathcal{S}(\tilde{\mathfrak{g}})_{\text{si}}$ is generated by the polynomials H_i defined above.*

Proof. First, we extract the “ \mathfrak{h} -part” out of $\tilde{\pi}$ and its powers. Let $\tilde{\pi}'$ be the Poisson tensor of $\tilde{\mathfrak{g}}'$ and ω' a volume form on $(\tilde{\mathfrak{g}}')^*$. Then $\tilde{\pi} = \tilde{\pi}' + R_{\mathfrak{h}}$, where $R_{\mathfrak{h}}$ is a sum $\sum [h_i, y_j] \partial_{h_i} \wedge \partial_{y_j}$ over a basis h_1, \dots, h_{ℓ} of \mathfrak{h} and a basis of $\tilde{\mathfrak{g}}'$. Since $\dim \mathfrak{h} = \ell$, for $k > \ell$, we have $\Lambda^k R_{\mathfrak{h}} = 0$. Taking into account that also $\Lambda^k \tilde{\pi}' = 0$ for $k > (n - 3\ell)/2$ (Lemma 5.7), we get the equality

$$\Lambda^{(n-\ell)/2} \tilde{\pi} = \left(\Lambda^{(n-3\ell)/2} \tilde{\pi}' \right) \wedge \left(\Lambda^{\ell} R_{\mathfrak{h}} \right).$$

Therefore a fundamental semi-invariant of $\tilde{\mathfrak{g}}'$ is a divisor of $p = \prod f_i^{r_i-1}$.

Set $\mathcal{H} := dH_1 \wedge \dots \wedge dH_{2\ell}$. By Lemma 5.7, $\mathcal{H} \neq 0$. Note also that by the same lemma, $\text{ind } \tilde{\mathfrak{g}}' = 2\ell$. Therefore, applying Lemma 2.1 to $\mathcal{S}(\tilde{\mathfrak{g}}')$, we get non-zero coprime $q_1, q_2 \in \mathcal{S}(\tilde{\mathfrak{g}}')$ such that

$$q_1(\mathcal{H}/\omega') = q_2 \Lambda^{(n-3\ell)/2} \tilde{\pi}'.$$

Since the polynomials q_1 and q_2 are coprime, q_1 must be a divisor of p as well. In particular, $\deg q_1 \leq \deg p$. Next we compute and sum the degrees of all objects involved in the equality

$$\begin{aligned} \deg q_2 + (n - 3\ell)/2 &= \deg q_1 + \deg \mathcal{H} \leq \deg p + \left(\sum_{i=1}^{\ell-1} \deg F_i - \ell + 1 \right) = \\ &= \deg p + ((n + \ell)/2 - \deg F_{\ell} - \ell + 1) = (n + \ell)/2 - (\ell + 1) - \ell + 1 = (n - 3\ell)/2. \end{aligned}$$

This is possible only if $q_2 \in \mathbb{K}$ and $q_1 = p$ (up to a scalar multiple). Thus $p(\mathcal{H}/\omega') = a \Lambda^{(n-3\ell)/2} \tilde{\pi}'$ with $a \in \mathbb{K}^{\times}$. Moreover, p is a fundamental semi-invariant of $\tilde{\pi}'$ and therefore the Jacobian locus $\mathcal{J}(H_1, \dots, H_{2\ell})$ of H_i does not contain divisors. We have $\dim \mathcal{J}(H_1, \dots, H_{2\ell}) \leq \dim \tilde{\mathfrak{g}} - 2$ and all polynomials H_i are homogeneous. This allows us to use the characteristic zero version of a result of Skryabin, see [PPY, Theorem 1.1], stating that here any $H \in \mathcal{S}(\tilde{\mathfrak{g}}')$ that is algebraic over a subalgebra generated by H_i is contained in that subalgebra. Since $\text{tr. deg } Z\mathcal{S}(\tilde{\mathfrak{g}}') \leq 2\ell$, we conclude that $Z\mathcal{S}(\tilde{\mathfrak{g}}')$ is generated by $H_1, \dots, H_{2\ell}$. Now the result follows from Lemma 5.6. \square

5.2. Subregular orbital varieties. Irreducible components of $\overline{\mathcal{O}^{\text{sub}} \cap \mathfrak{n}}$ are called *subregular orbital varieties*. They have played a major rôle in the proof of Theorem 5.5 and we know that each of them is a linear space \mathfrak{D}_i . Every \mathfrak{D}_i is also the nilpotent radical of a minimal parabolic subalgebra \mathfrak{p}_{α_i} . An interesting question is whether B acts on \mathfrak{D}_i with an open orbit. This problem was addressed and solved in [GHR]. As it turns out, our results complement and simplify some of the arguments in [GHR].

Let $\tilde{G} = B \ltimes \exp(\mathfrak{n}^-)$ be an algebraic group with $\text{Lie } \tilde{G} = \tilde{\mathfrak{g}}$. Let also \mathcal{O}_i be an irreducible component of $\mathcal{O}^{\text{sub}} \cap \mathfrak{n}$ lying in \mathfrak{D}_i . Note that \mathcal{O}_i is a dense open subset of \mathfrak{D}_i .

Lemma 5.9. *Suppose that $r_i = [\delta : \alpha_i] = 1$. Then there is a dense open B -orbit in \mathfrak{D}_i .*

Proof. According to Theorem 5.5, if $r_i = 1$, then the intersection $(\mathfrak{b}^* \times \mathfrak{D}_i) \cap \tilde{\mathfrak{g}}_{\text{reg}}^*$ is non-empty, and hence it is a non-empty open subset of $\mathfrak{b}^* \times \mathfrak{D}_i$. Thereby there is a regular x in $\mathfrak{b}^* \times \mathfrak{O}_i$. Let \hat{e} be the \mathfrak{n}^{ab} -component of this x . In other words, $x \in \mathfrak{b}^* \times \{\hat{e}\}$, where $\hat{e} \in \mathfrak{n}^{\text{ab}}$ comes from a subregular nilpotent element $e \in \mathfrak{n}$. Next we compute the codimension of $\tilde{G}x$. This can be done in the spirit of the Raïs formula for the index of a semi-direct product [R], see also [P07', Proposition 5.5] and Lemma 4.8. And the result is that

$$(5.1) \quad \dim \tilde{\mathfrak{g}} - \dim \tilde{G}x = \dim \mathfrak{n} - (\dim \mathfrak{b} - \dim \mathfrak{b}_e) + \text{ind } \mathfrak{b}_e = \dim \mathfrak{b}_e - \ell + \text{ind } \mathfrak{b}_e.$$

Suppose that Be is not dense in \mathfrak{D}_i . Then $\dim Be < (\dim \mathfrak{n} - 1)$ and therefore $\dim \mathfrak{b}_e \geq \ell + 2$ implying $\mathfrak{b}_e = \mathfrak{g}_e$. Since $\text{ind } \mathfrak{g}_e \geq \ell$ by Vinberg's inequality, [P03, Corollary 1.7], we get that $\dim \tilde{\mathfrak{g}} - \dim \tilde{G}x \geq (\ell + 2)$ and therefore $x \in \tilde{\mathfrak{g}}_{\text{sing}}^*$. This contradiction proves the lemma. \square

In type A all r_i are equal to 1 and all components \mathfrak{D}_i have open B -orbits. This result was first obtained by J.A. Vargas [V].

Example 5.10. Suppose that $\mathfrak{g} = \mathfrak{so}_{2\ell}$ with $\ell > 3$ and $e \in \mathfrak{g}$ is a subregular nilpotent element. Then e is given by a partition $(2\ell - 3, 3)$, odd powers e^{2k+1} of the underlying matrix are elements of \mathfrak{g} , and \mathfrak{g}_e has a basis

$$e, e^3, \dots, e^{2\ell-5}, \xi_1, \xi_2, \xi_3, \eta$$

with the non-trivial commutators: $[\xi_1, \xi_2] = e^{2\ell-5}$, $[\xi_1, \eta] = \xi_2$, and $[\xi_2, \eta] = \xi_3$. (The structure of \mathfrak{g}_e is described, for example, in [Y, Section 1].) It is not difficult to see that \mathfrak{g}_e does not contain a commutative subalgebra of codimension 1. Coming back to equation (5.1), we get that in case $\dim \mathfrak{b}_e = \ell + 1$, the stabiliser \mathfrak{b}_e is not commutative and thereby $\text{ind } \mathfrak{b}_e \leq \ell - 1$. As a consequence, $\dim \tilde{G}x \geq \dim \tilde{\mathfrak{g}} - \ell$. Thus in type D there is an open B -orbit in \mathfrak{D}_i if and only if $r_i = 1$.

Calculating centralisers \mathfrak{g}_e of subregular nilpotent elements in type E , on GAP or by hand, one can show that \mathfrak{g}_e does not contain an Abelian subalgebra of codimension 1. Together with Theorem 5.5 and equation (5.1), this fact provides an additional explanation for [GHR, Theorem 2.4(a)(i)]. That result states that for \mathfrak{g} simply laced, \mathfrak{D}_i contains an open B -orbit if and only if $r_i = 1$.

Remark 5.11. Actually, Theorem 2.4 of [GHR] asserts that there is a finite number of B -orbits in \mathfrak{O}_i , if $r_i = 1$. As is explained in the Introduction of [GHR], this is equivalent to the existence of an open orbit. Note also that [GHR] proves the existence of an open B -orbit by giving its representative in each particular case. Besides, results for the exceptional Lie algebras rely on GAP calculations of S. Goodwin [G].

There is another interesting related question, asked by D. Panyushev. When is the stabiliser B_e of a generic $e \in \mathfrak{D}_i$ Abelian? The centraliser of a nilpotent element is Abelian only when the element is regular, for a conceptual proof of this fact see [P03, Theorem 3.3]. In particular, \mathfrak{g}_e is not Abelian for a subregular nilpotent element e . This implies that \mathfrak{b}_e can be Abelian only if $\dim \mathfrak{b}_e < \ell + 2$ and there is a dense B -orbit in \mathfrak{D}_i . On the other hand, for an Abelian Lie algebra, $\dim \mathfrak{b}_e = \dim \mathfrak{b}_e = \ell + 1$ and therefore r_i must be larger than 1. In the simply laced case \mathfrak{g}_e does not contain Abelian subalgebras of codimension 1. For the remaining Lie algebras, [GHR, Theorem 2.4.(a)(ii)] provides the following answer.

Proposition 5.12. *The stabiliser in B of a generic $e \in \mathfrak{D}_i$ is Abelian if and only if*

- \mathfrak{g} is of type B_ℓ and $i > 1$, or
- \mathfrak{g} is of type C_ℓ with $\ell > 1$ and $i = 1$, or
- \mathfrak{g} is of type F_4 and $i = 4$, or
- \mathfrak{g} is of type G_2 and $i = 2$,

in the Vinberg-Onishchik numbering of simple roots [VO].

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